CS103
FALL 2025

Lecture 07:

Functions

Part 2 of 2

Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.

Recap from Last Time

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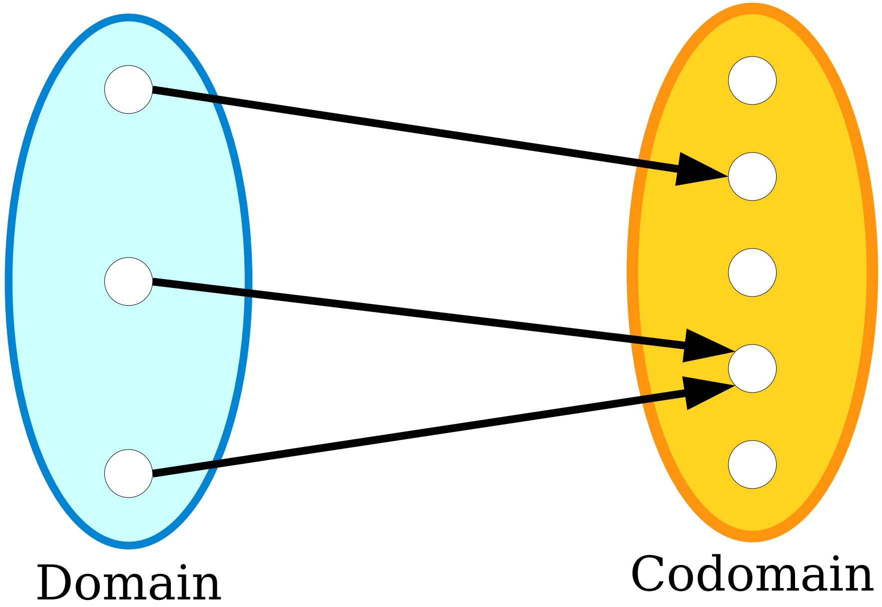
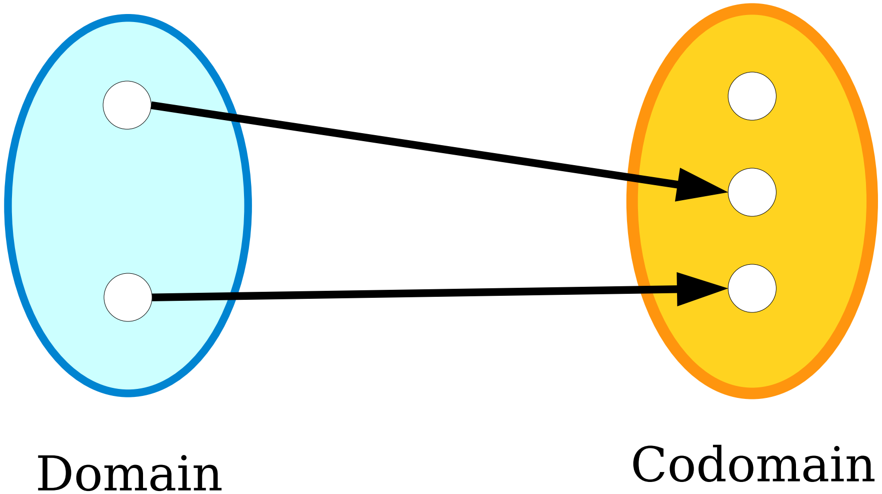


Is it a function? **Yes!**

Is it an injection? **No.**

Is it a surjection? **No.**

Recap from Last Time



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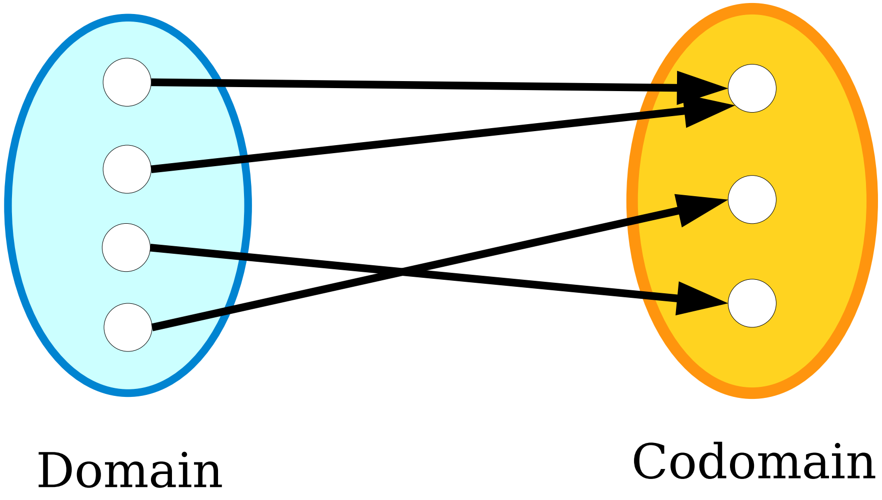


Is it a function? **Yes!**

Is it an injection? **No.**

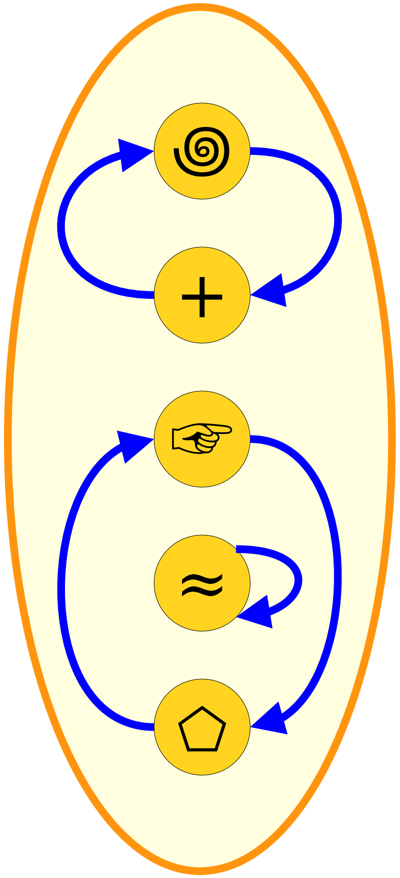
Is it a surjection? **Yes!**

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** when the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")



- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.

		To ***prove*** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$		Assume A is true, then prove B is true.
$A \land B$		Prove A. Also prove B.
$A \lor B$		Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \to B$ and $B \to A$.
$\neg A$		Simplify the negation, then consult this table on the result.

New Stuff!

A Proof About Birds



Theorem: If all birds have feathers, then all herons have feathers.

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Given the predicates

Bird(b), which says b is a bird;
Heron(h), which says h is a heron; and
Feathers(x), which says x has feathers,

translate the theorem into first-order logic.

$$(\forall b.\ (Bird(b) \to Feathers(b))) \to (\forall h.\ (Heron(h) \to Feathers(h)))$$

All birds have feathers

All herons have feathers

Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

Consider an arbitrary bird *b*. Since *b* is a bird, *b* has feathers. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

$$\underbrace{(\forall b.\ (Bird(b) \to Feathers(b)))}_{\text{All birds have feathers}} \to \underbrace{(\forall h.\ (Heron(h) \to Feathers(h)))}_{\text{All herons have feathers}}$$

Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

Consider an arbitrary heron h. We will show that h has feathers. To do so, note that since h is a heron we know h is a bird. Therefore, by our earlier assumption, h has feathers. ■

$$\underbrace{(\forall b.\ (Bird(b) \to Feathers(b)))}_{\text{All birds have feathers}} \to \underbrace{(\forall h.\ (Heron(h) \to Feathers(h)))}_{\text{All herons have feathers}}$$

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds have feathers.
  - Here, we ***proved*** all herons have feathers.
- Statements behave differently based on whether you're assuming or proving them.

$$\underbrace{(\forall b.\,(Bird(b) \to Feathers(b)))} \to \underbrace{(\forall h.\,(Heron(h) \to Feathers(h)))}$$

We never introduce a variable *b*.

We introduce a variable *h* almost immediately.

Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$(\forall b.\ (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h.\ (Heron(h) \rightarrow Feathers(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.

- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, have feathers.

$$(\forall b.\ (Bird(b) \to Feathers(b))) \to (\forall h.\ (Heron(h) \to Feathers(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

	If you ***assume*** this is true…	To ***prove*** that this is true…
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We now have three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

f is an involution.

f is surjective.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$

Assume this.

Prove this.

If you ***assume*** this is true…

Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

Assume this.

Prove this.

Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$

Ass

There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

Now, we hit an existential quantifier. Since we're *proving* this, we need to find a choice of $a \in A$ where this is true.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

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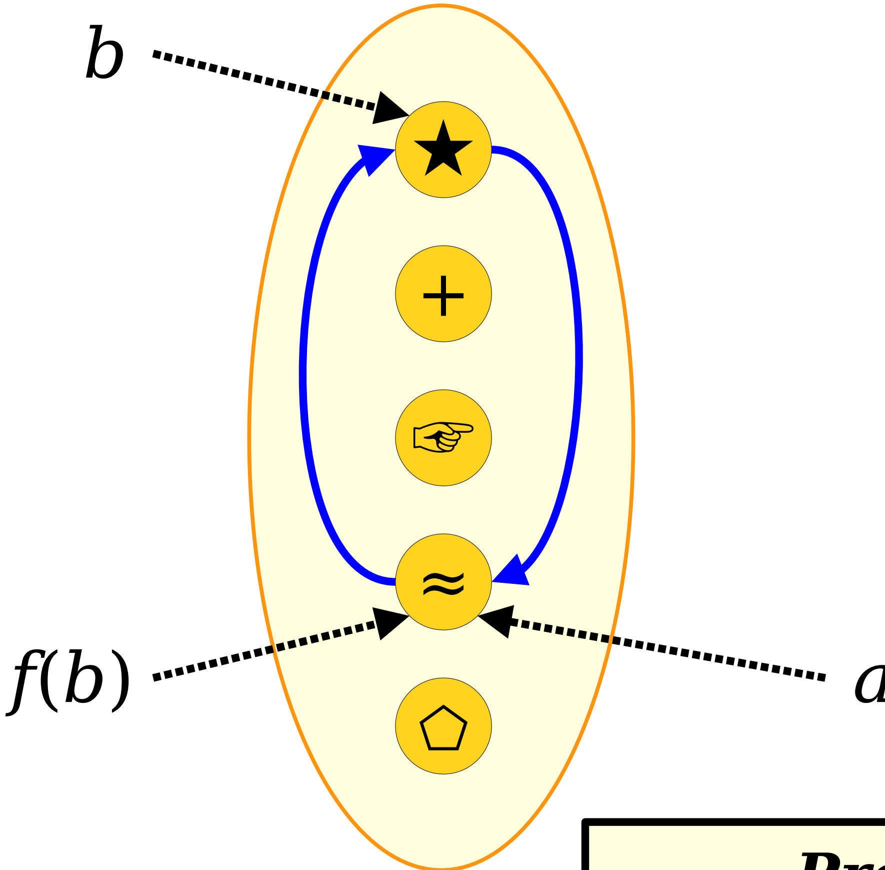

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Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ■

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

The Two-Column Proof Organizer

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

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What We're Assuming	***What We Need to Prove***
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming	***What We Need to Prove***
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	~~$\forall a_1 \in A.\ \forall a_2 \in A.$~~ $(f(a_1) = f(a_2) \to a_1 = a_2)$
$a_1 \in A$	
$a_2 \in A$	

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$a_1 \in A$	
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We need to prove this implication. So we assume the antecedent and prove the consequent.

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming	What We Need to Prove
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	~~$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to$~~ $a_1 = a_2$ ~~$)$~~
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$a_2 \in A$	
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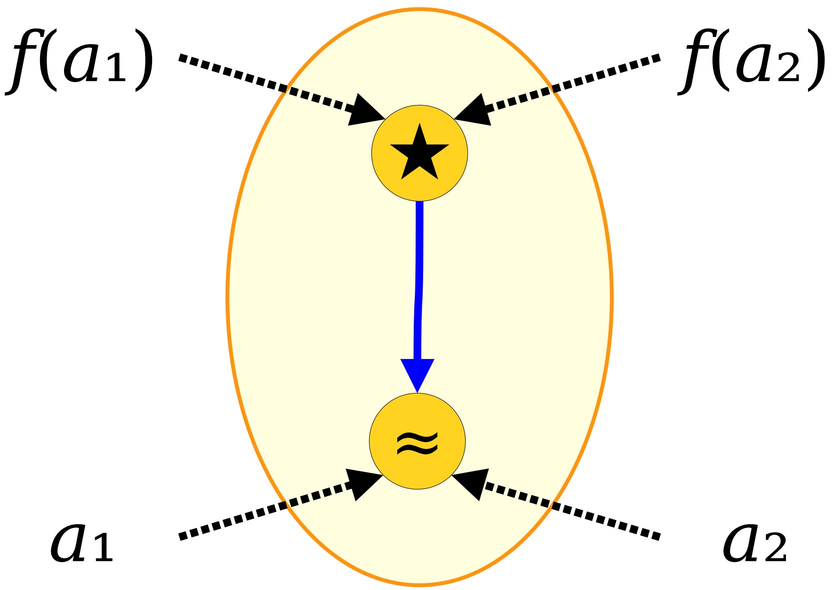


Theorem: Let $f : A \to A$ be an involution. Then f is injective.

Proof: Choose any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We need to show that $a_1 = a_2$.

Since $f(a_1) = f(a_2)$, we know that $f(f(a_1)) = f(f(a_2))$. Because f is an involution, we see $a_1 = f(f(a_1))$ and that $f(f(a_2)) = a_2$. Putting this together, we see that

$$a_1 = f(f(a_1)) = f(f(a_2)) = a_2,$$

so $a_1 = a_2$, as needed. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

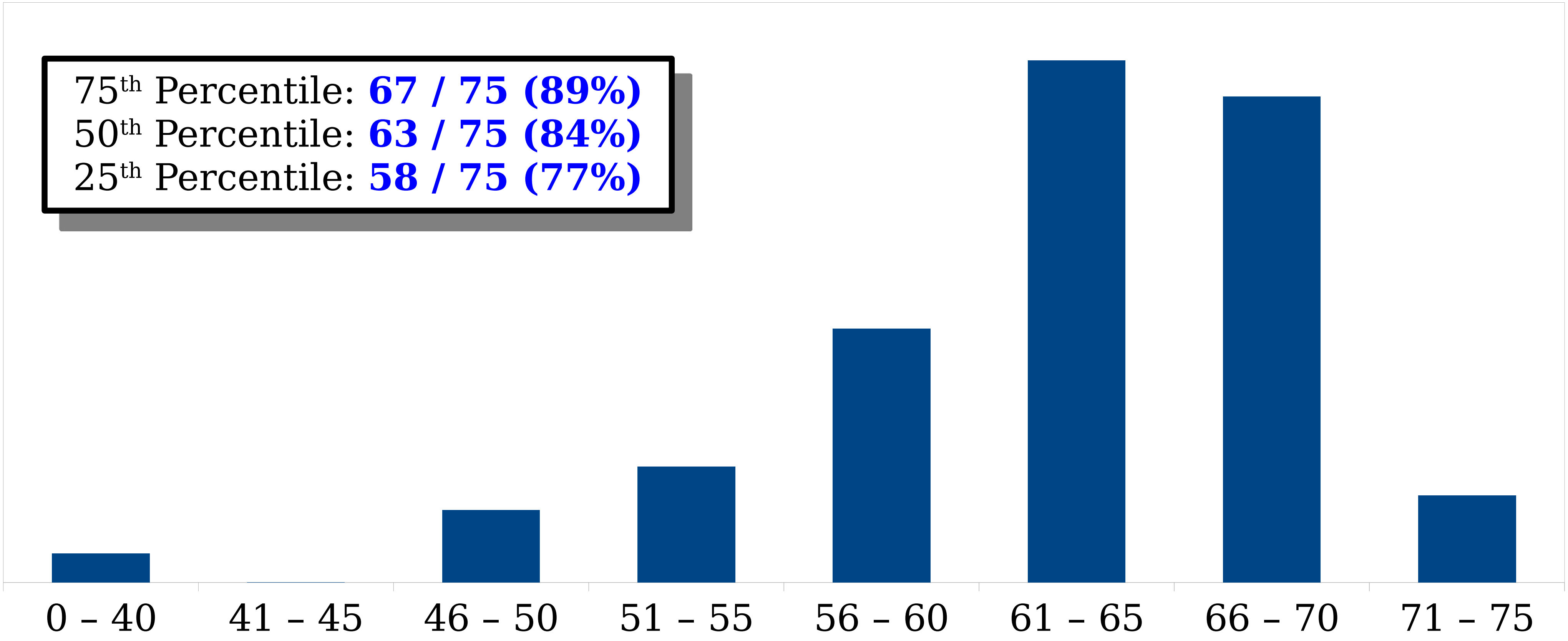
Time-Out for Announcements!

Problem Set One Graded

- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set One Graded

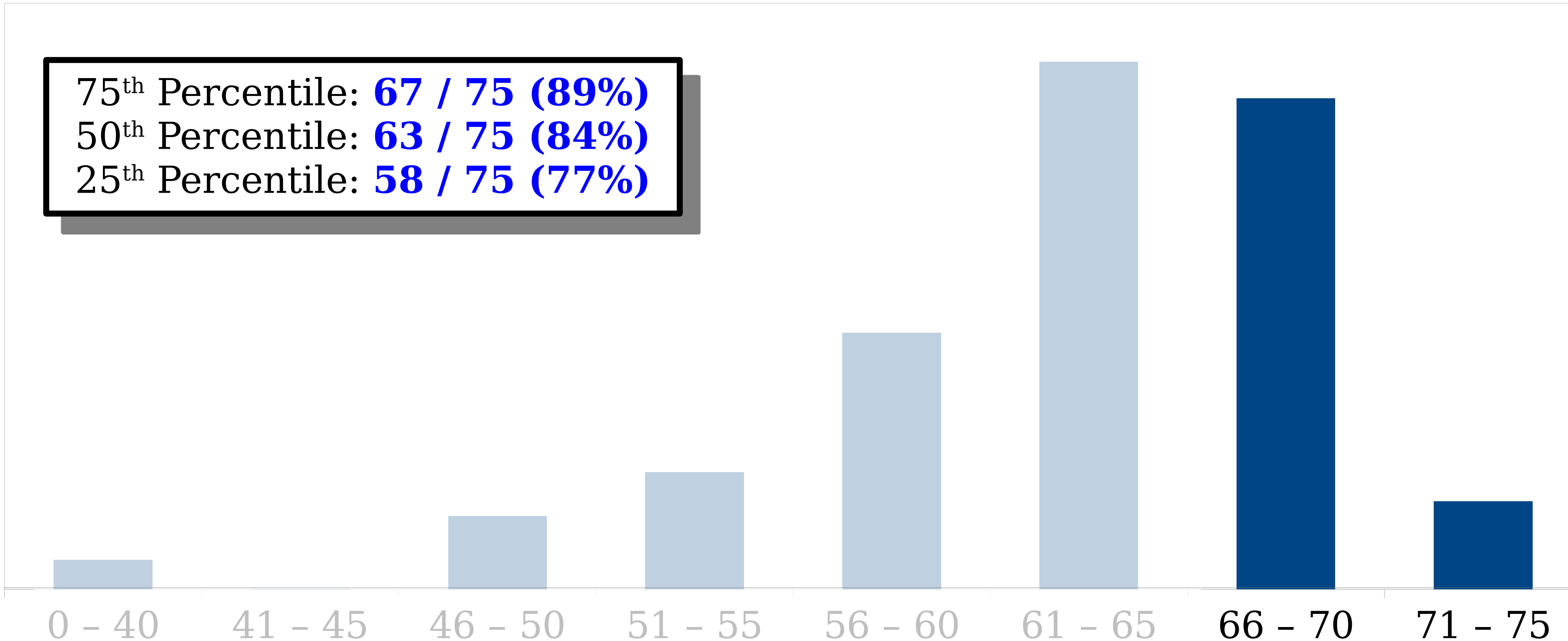
75th Percentile: **67 / 75 (89%)**
50th Percentile: **63 / 75 (84%)**
25th Percentile: **58 / 75 (77%)**



Pro tips when reading a grading distribution:

1. Standard deviations are ***unhelpful and discouraging***. Ignore them.
2. The average score is a ***unhelpful***. Ignore it.
3. Raw scores are ***unhelpful and discouraging***. Ignore them.

Problem Set One Graded



"Great job! Look over your feedback for some tips on how to tweak things for next time."

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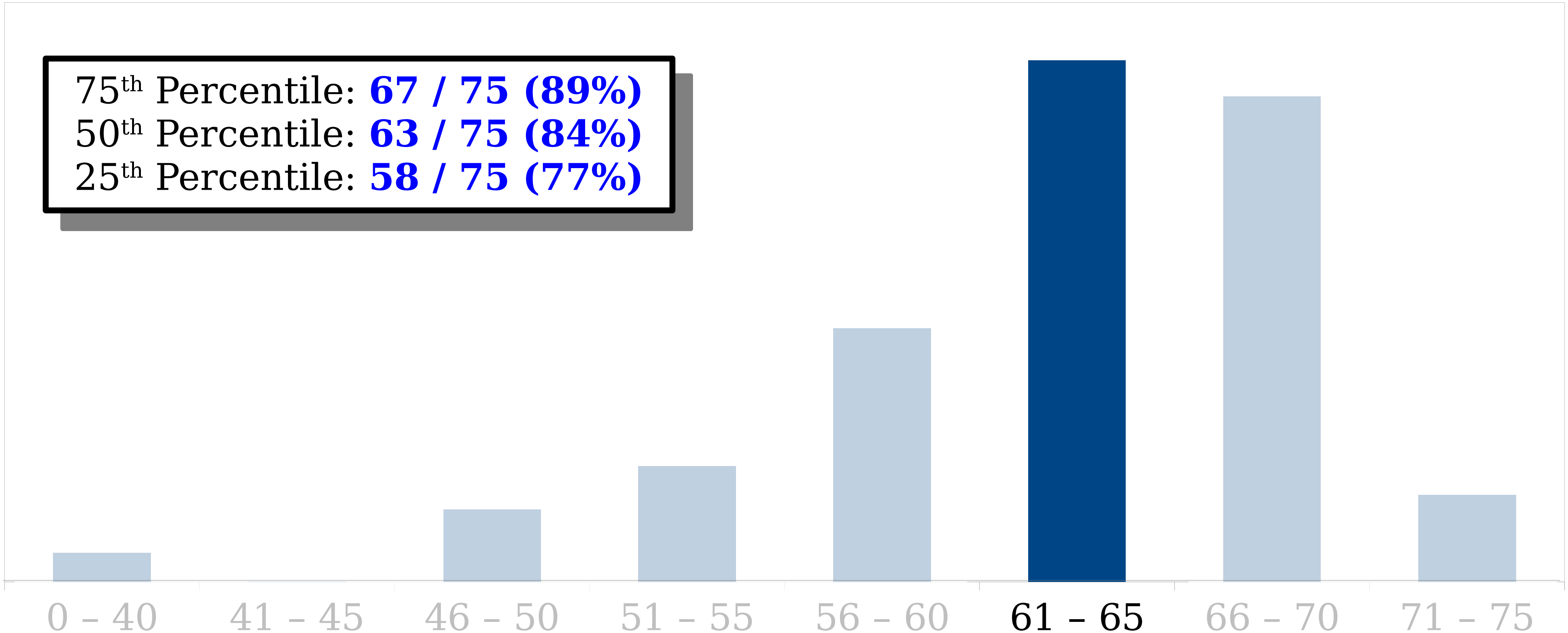

"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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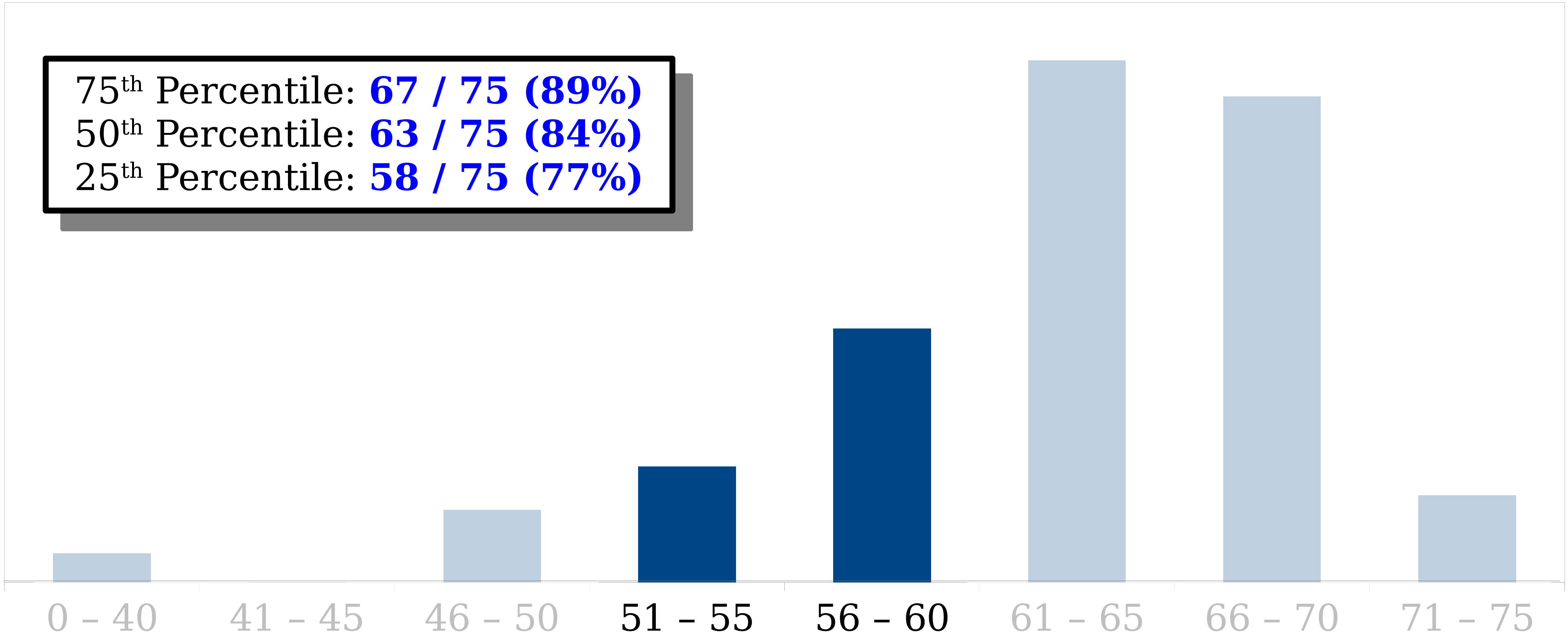
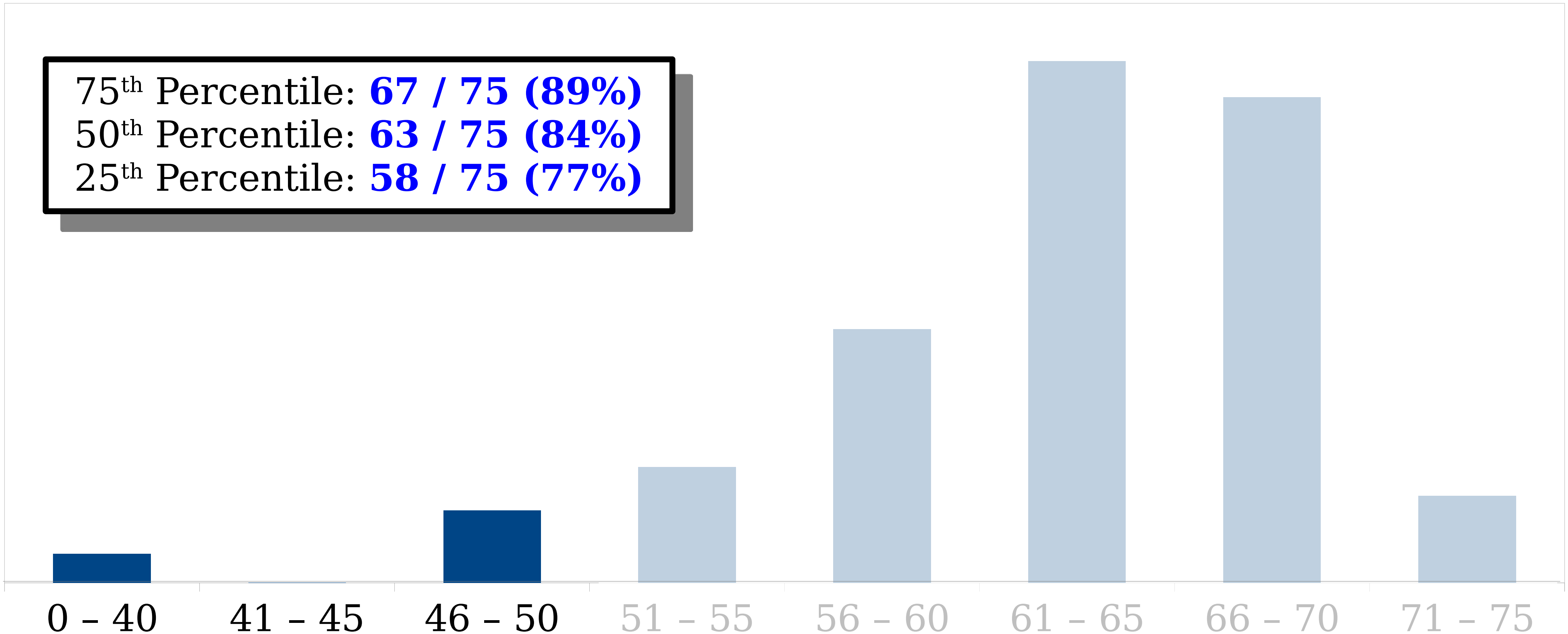


"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded



"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged – you can do this!"

What Not to Think

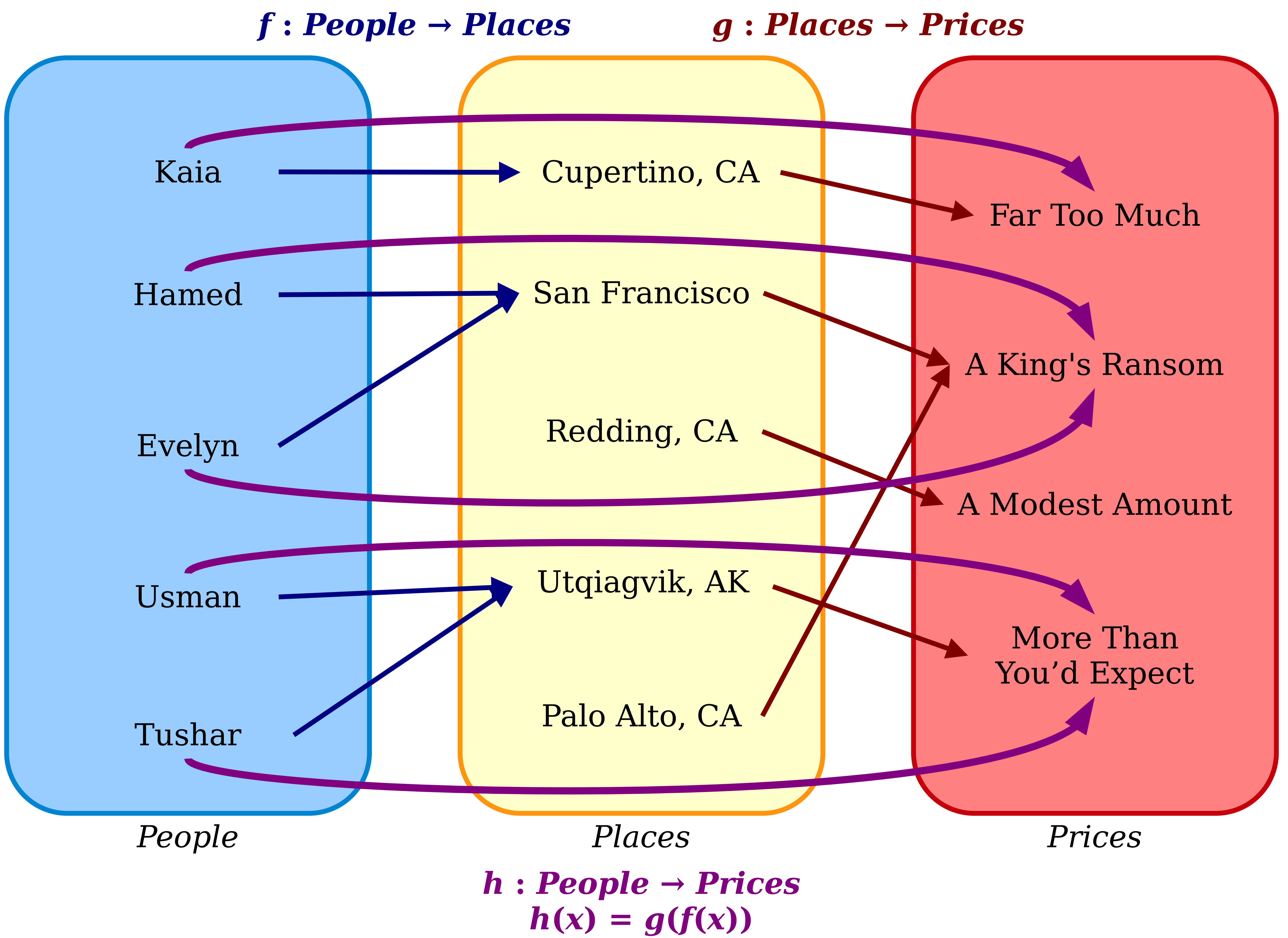
- "Well, I guess I'm just not good at math."
  - For most of you, this is your first time doing proof-based math.
  - It is ***totally normal*** when learning any new skill to have areas where you need to improve. And we cover a ton of material here!
  - You will improve over the quarter. Hang in there!
- "I got a good score, so I don't need to review anything."
  - Check your feedback. Make sure you didn't miss an important detail.
  - We let you work in pairs. Be honest with yourself – did you lean too much on your partner? Could you have done the work unassisted?
  - We provide lots of office hours. Be honest with yourself – did you get too much help from the TAs?
- You will need to be able to solve problems like these solo on the exams. ***Put in the time now to patch up any gaps in your understand***

Essential Action Items

- ***Review your feedback.***
  - Don’t just look at the raw score. Make sure you really, truly understand where you need to improve.
- ***Read the solutions in depth.***
  - Make sure you understand what we were asking, why we asked it, and what we wanted you to take away.
  - (Especially for Q8, Q10) Look at our solutions and see if there’s any neat lessons you can draw from them.
- ***Come to us with questions.***
  - Anything you’re not sure about? That’s what we’re here for! Come to office hours, ask questions on EdStem, etc.

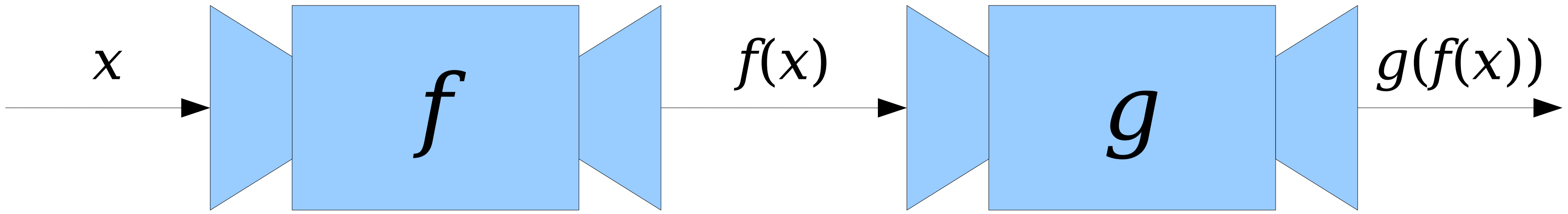
Back to CS103!

Function Composition


f : People → Places
g : Places → Prices
Kaia
Hamed
Evelyn
Usman
Tushar
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than You'd Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.
- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is g's codomain.
  - Even though composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.
- Composition is ***associative***: $(f \circ g) \circ h = f \circ (g \circ h)$. (Prove this!)
- Composition is not necessarily commutative: $f \circ g$ is not necessarily the same as $g \circ f$. (Prove this!)

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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$a_1 \in A$ is arbitrarily-chosen.

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$g : B \to C$ is an injection.	
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

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$g : B \to C$ is an injection.	Let's write this out separately and simplify things a bit.
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$ $g(x) \neq g(y)$ $)$	
$a_1 \in A$ is arbitrarily-chosen.	
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$g : B \to C$ is an injection.	$g(f(a_1)) \neq g(f(a_2))$
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$ $g(x) \neq g(y)$ $)$	
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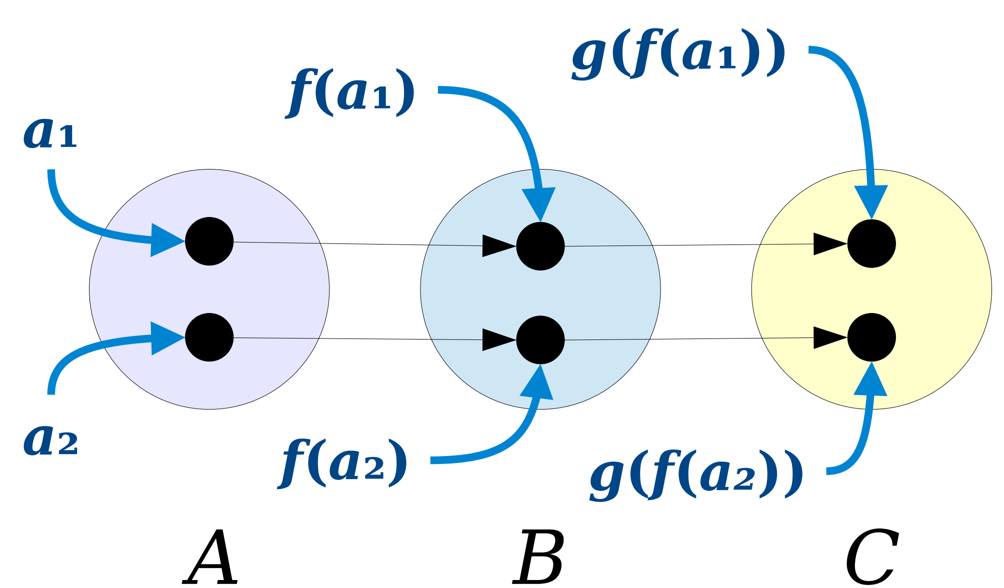
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

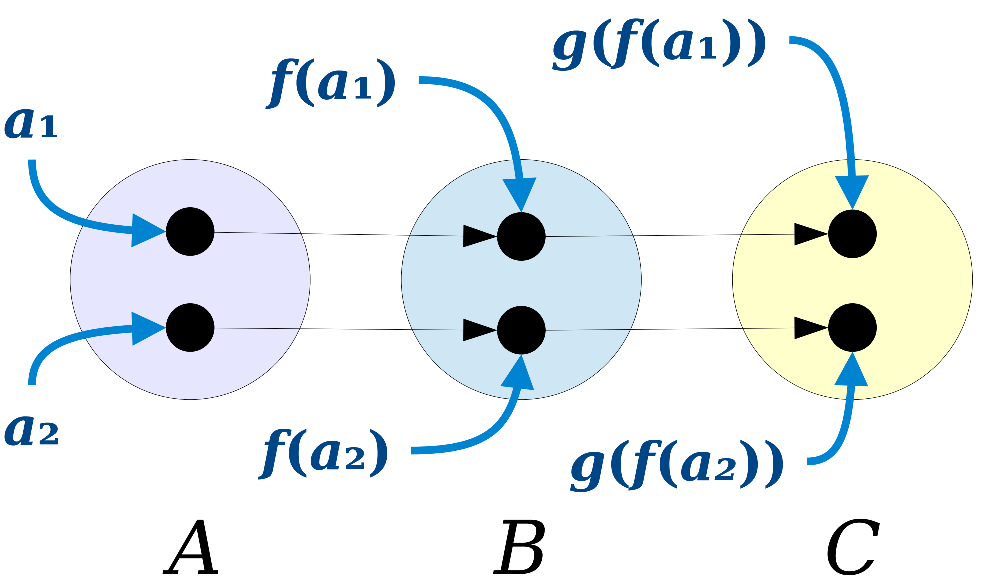


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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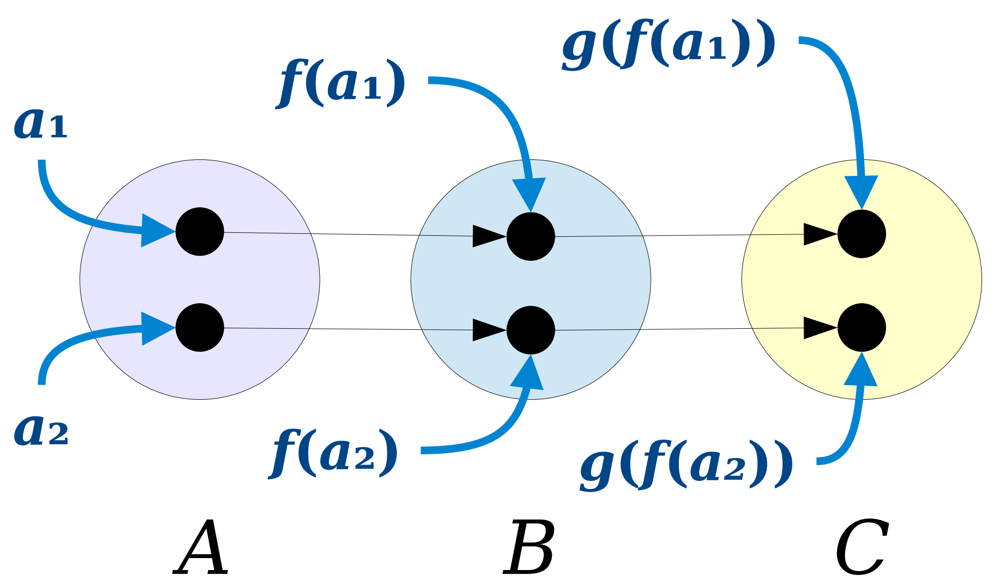
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: In the appendix!

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	If you ***assume*** this is true…	To ***prove*** that this is true…
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Set Theory Revisited***
  - Formalizing our definitions.
- ***Proofs on Sets***
  - How to rigorously establish set-theoretic results.

Appendix: Additional Function Proofs

Proof: Composing surjections yields a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■